A direct approach for optimal control of linear delay systems

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Abstract

This paper presents a numerical method for solving linear optimal control problem of delay systems. This method is based on orthogonal triangular functions and their structural properties. The differential and integral expressions which arise in the delay systems and the performance index are converted into some algebraic equations which can be solved for the unknown coefficients. An illustrative example is included to demonstrate the validity and accuracy of the technique.

Key words: Delay Systems, Optimal Control, Performance Index, Orthogonal Triangular Functions.

1. Introduction

The system with time delay have been of considerable concern. There are many practical systems like population growth, epidemic, neural networks appropriately modeled by delay-differential equations. Delays appears frequently in population ecology, disturbed networks containing lossless transmission lines, heat exchangers, robots in contact with rigid environments, etc, [7]. Time delay systems are a very important class of systems whose control and optimization have been of interest to many investigators. There are important aspects of delay of differential equations, partial differential equations and integral equations which have been studied in [2] and references therein. Two types of delay exist, the first is in the states and the second is in both states and inputs. Kharatishvili, [5] and Krasovshii, [6] have provided the basic foundation. The numerical methods to solve the delay systems using orthogonal functions have been presented by [10] used block-pulse and the hybrid functions of block-pulse functions plus Legendre polynomials operational matrices of integration, respectively, to calculate the integral involved in the performance index. Deb *et al.* in [1] have introduced a complementary pair of orthogonal triangular function (TF) sets from the well-known block pulse function (BPF) set**.** They have used theses functions to analysis the solution of dynamic systems. Maleknejad *et al.* have applied TFs to obtain the solution of integral equations and the optimal control of them in [8,9]. In the present paper, we first briefly review of TF s and their properties, then we apply a direct method based upon TF s to approximate the optimal control of time-delay systems and the performance index of them. The operational matrices of product, delay and integration have been used to reduce the solution of optimal control of linear time-delay systems to the solution of algebraic equations. We will demonstrate the numerical results by considering an illustrative.

2.Briefly review of TF s

Definition 1. A set of BPF, $\mathbf{\Psi}(t)$ containing m component functions in the semi-open interval $[0,T)$ is given by

$$\mathbf{\Psi(t)} = [\psi_0(t) \quad \psi_1(t) \quad \dots \quad \psi_i(t) \quad \dots \quad \psi_{m-1}(t)]^T, \qquad (1)$$

where $[...]^T$ denotes transpose.

The i th component $\psi_i(t)$ of the BPF vector $\mathbf{\Psi}(t)$ is defined as

$$\psi_i(t) = \begin{cases} 1 \quad , & \frac{iT}{m} \le t < \frac{(i+1)T}{m}, \\ 0 \quad , & otherwise, \end{cases} \qquad (2)$$

where $i = 0,1,2,...,(m-1)$ and $h = \frac{T}{m}$, [1].

2.1 BPF expansion of functions

A function $x(t) \in L^2[0,T)$ can be expanded in terms of BPF s as follows

$$x(t) \approx [x_0 \quad x_1 \quad x_2 \quad \dots \quad x_i \quad \dots \quad x_{m-1}]\, \mathbf{\Psi(t)} = X^T \mathbf{\Psi}(t), \qquad (3)$$

where the constant coefficients x_i's are given by

$$x_i = (\frac{1}{h}) \int_{ih}^{(i+1)h} x(t)\, dt. \qquad (4)$$

Definition 2. Let $\psi_i(t)$ be the i th member of an m-set BPF, we have dissected a BPF into two TF s as

$$\psi_i(t) = T1_i(t) + T2_i(t), \qquad (5)$$

where $T1_i(t)$ and $T2_i(t)$ are the i th components of the $m-$vectors $T1(t)$ and $T2(t)$, which are defined in [Deb, Cnss] as follows**:**

$$T1_i(t) = \begin{cases} 1 - \frac{(t-ih)}{h}, & ih \le t < (i+1)h \\ 0, & otherwise, \end{cases} \qquad (6)$$

and

$$T2_i(t) = \begin{cases} \frac{(t-ih)}{h}, & ih \le t < (i+1)h \\ 0, & otherwise, \end{cases} \qquad (7)$$

for $i = 0,1,2,...,(m-1)$.

2.2 TF expansion of functions

(**i**) Suppose $x(t)$ be an $L^2[0,T)$, the expansion of x in terms of TF s is as the following

$$\begin{aligned} x(t) &\approx \sum_{i=0}^{m-1} \left[x_i T1_i(t) + x_{i+1} T2_i(t) \right] \\ &= \left[x_0 \quad x_1 \quad \dots \quad x_i \quad \dots \quad x_{m-1} \right] \mathbf{T1}(t) + \left[x_1 \quad x_2 \quad \dots \quad x_i \quad \dots \quad x_m \right] \mathbf{T2}(t) \\ &= X_1^T \, \mathbf{T1}(t) + X_2^T \, \mathbf{T2}(t). \end{aligned} \tag{8}$$

where the sequence of coefficients $\{x_i\}_{i=0}^{m}$ are given by

$$x_i = x(ih), \quad i = 0,1,\dots,m. \tag{9}$$

(**ii**) Similarly, suppose $y(t)$ be an L^2 function on $[0,T)$, by expanding $X(t)$ and $y(t)$ in terms of TF s, we get

$$x(t)y(t) = X_1^T \hat{Y}_1 \, \mathbf{T1}(t) + X_2^T \hat{Y}_2 \, \mathbf{T2}(t), \tag{10}$$

where $\hat{Y}_1$ and $\hat{Y}_2$ are diagonal $m \times m$ matrices and we have

$$\hat{Y}_1 = \begin{pmatrix} y_0 & 0 & \dots & 0 \\ 0 & y_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_{m-1} \end{pmatrix}, \qquad \hat{Y}_2 = \begin{pmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_m \end{pmatrix}. \tag{11}$$

2.3 Multiplication of TF s

It is obviously that

$$\mathbf{T1}(t)\mathbf{T2}^T(t) \approx 0, \qquad \mathbf{T2}(t)\mathbf{T1}^T(t) \approx 0.$$

Also,

$$\begin{aligned} \int_0^1 \mathbf{T1}(t)\mathbf{T1}^T(t)\,dt &= \int_0^1 \mathbf{T2}(t)\mathbf{T2}^T(t)\,dt \approx \frac{h}{3} I, \\ \int_0^1 \mathbf{T1}(t)\mathbf{T2}^T(t)\,dt &= \int_0^1 \mathbf{T2}(t)\mathbf{T1}^T(t)\,dt \approx \frac{h}{6} I, \end{aligned} \tag{12}$$

where I is $m \times m$ identity matrix, [8].

2.4 Operational matrices of integration

The operational matrices of integration for TF s have been given by Deb in [1], as follows

$$\int_0^t \mathbf{T1}(\tau)\,d\tau = \int_0^t \mathbf{T2}(\tau)\,d\tau = P_1 \mathbf{T1}(t) + P_2 \mathbf{T2}(t), \tag{13}$$

where P_1 and P_2 are $m \times m$ matrices as follows

$$P_1 = \frac{h}{2} \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad P_2 = \frac{h}{2} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{14}$$

2.5 Delay operational matrices of TF s

The delay function $\mathbf{T}1(t-\tau)$ and $\mathbf{T2}(t-\tau)$ are the shift of functions $\mathbf{T}1(t)$ and $\mathbf{T2}(t)$ along the time axis by τ. We can define

$$\mathbf{T1}(t-\tau)=D_{T1}\ \mathbf{T1}(t),$$
$$\mathbf{T2}(t-\tau)=D_{T2}\ \mathbf{T2}t), \tag{15}$$

for $t>\tau,\ 0\le t\le T,$ where D_{T1} and D_{T2} are TF's delay operational matrices which can be obtained as the following

$$D_{T1}=D_{T2}=\begin{bmatrix} 0 & \vdots & I_{(m-1)\times(m-1)} \\ \dots & \dots & \dots\dots \\ 0 & \vdots & 0 \end{bmatrix}, \qquad D^i\ge 0,\ i\ge m. \tag{16}$$

3. Problem statement

Consider a linear time-varing delay system as follows

$$\dot{X}(t)=E(t)X(t)+F(t)X(t-\tau)+G(t)U(t), \qquad 0\le t\le t_f, \tag{17}$$
$$X(0)=X_0,$$
$$X(t)=\varphi(t), \qquad -\tau\le t<0,$$

where $X(t)\in R^n, U(t)\in R^m, E(t), F(t)$ and $G(t)$ are matrices of appropriate dimensions, X_0 is a constant specified vector and $\varphi(t)$ is an arbitrary known function. The problem is to find the optimal control $X(t)$ and $U(t)$, which minimizing the quadratic cost functional

$$J=\frac{1}{2}X^T(T)\,S\,X(T)+\frac{1}{2}\int_0^T\left[X^T(t)\,Q(t)\,X(t)+U^T(t)\,R(t)U(t)\right]dt, \tag{18}$$

where $S, Q(t)$ and $R(t)$ are matrices of appropriate dimensions with S and $Q(t)$ symmetric positive semi-definite matrices and $R(t)$ a symmetric positive definite matrix. The purpose is to obtain the optimal control which under the above conditions of fixed initial state and terminal time and free final state, minimizes J subject to Eq. (18).

4. Approximation using triangular functions

Using Eq. (8), $X(t)$, $U(t)$, X_0 and $\varphi(t)$ in Eq. (17) can be approximated in terms of TF s as

$$X(t)=X_1^T\,\mathbf{T1}(t)+X_2^T\,\mathbf{T2}(t),$$
$$U(t)=U_1^T\,\mathbf{T1}(t)+U_2^T\,\mathbf{T2}(t),$$
$$X_0=X_{10}^T\,\mathbf{T1}(t)+X_{20}^T\,\mathbf{T2}(t),$$
$$\varphi(t)=\Phi_1^T\,\mathbf{T1}(t)+\Phi_2^T\,\mathbf{T2}(t), \tag{19}$$

where $X_1,\ X_2, U_1, U_2,\ X_{10},\ X_{20},\ \Phi_1$ and Φ_2 are $m-$vectors.

By using Eq. (16), we also have

$$X(t-\tau)=\begin{cases} \Phi_1^T\,\mathbf{T1}(t)+\Phi_2^T\,\mathbf{T2}(t) & 0\le t\le\tau, \\ X_1^T D_{T_1}\mathbf{T1}(t)+X_2^T D_{T_2}\mathbf{T2}(t), & \tau\le t\le T. \end{cases} \tag{20}$$

Moreover, by using Eq. (11), we have

$$E(t)X(t)=X_1^T\hat{E}_1\,\mathbf{T1}(t)+X_2^T\hat{E}_2\,\mathbf{T2}(t),$$
$$G(t)U(t)=U_1^T\hat{G}_1\,\mathbf{T1}(t)+U_2^T\hat{G}_2\,\mathbf{T2}(t), \tag{21}$$

where $\hat{E}_1, \hat{E}_2, \hat{G}_1$ and $\hat{G}_2$ are diagonal matrices such that their diagonal matrices such that their elements are the elements of E_1, E_2, G_1 and G_2, respectively. Similarly, using Eqs. (11), (13) and (20), we get

$$\int_0^t F(t')\,X(t'-\tau)dt' \approx \begin{cases} L_1^T\,\mathbf{T1}(t)+L_2^T\,\mathbf{T2}(t), & 0\le t\le\tau, \\ L_3^T\,\mathbf{T1}(t)+L_4^T\,\mathbf{T2}(t), & \tau\le t\le T, \end{cases} \tag{22}$$

where

$$L_1 = P_1^T(\hat{F}_1\Phi_1+\hat{F}_2\Phi_2),$$
$$L_2 = P_2^T(\hat{F}_1\Phi_1+\hat{F}_2\Phi_2),$$
$$L_3 = P_1^T(\hat{F}_1D_{T1}^TX_1+\hat{F}_2D_{T2}^TX_2)+Z_1^T(\hat{F}_1\Phi_1+\hat{F}_2\Phi_2),$$
$$L_4 = P_2^T(\hat{F}_1D_{T1}^TX_1+\hat{F}_2D_{T2}^TX_2)+Z_2^T(\hat{F}_1\Phi_1+\hat{F}_2\Phi_2),$$

and Z_1 and Z_2 are constant matrices of order $m\times m$, which can be obtained as follows

$$\int_0^\tau \mathbf{T1}(t)\,dt = \int_0^\tau \mathbf{T2}(t)\,dt \approx Z_1\,\mathbf{T1}(t)+Z_2\mathbf{T2}(t). \tag{23}$$

By integrating Eq. (17) from 0 to t and using Eqs. (19)~(23), we get

$$\omega_1^*(X,U) = [I-P_1^T\,\Omega_1]X_1 - P_1^T\,\Omega_2\,X_2 - P_1^T\hat{G}_1U_1 - P_1^TG_2U_2 - \gamma_1 = 0,$$
$$\omega_2^*(X,U) = -P_2^T\,\Omega_1\,X_1 + [I-P_2^T\,\Omega_2]X_2 - P_2^T\hat{G}_1U_1 - P_2^TG_2U_2 - \gamma_2 = 0, \tag{24}$$

where

$$\Omega_1 = \hat{E}_1 + \hat{F}_1D_{T1}^T,$$
$$\Omega_2 = \hat{E}_2 + \hat{F}_2D_{T2}^T,$$
$$\gamma_1 = (P_1^T+Z_1^T)(\hat{F}_1\Phi_1+\hat{F}_2\Phi_2)+X_{10},$$
$$\gamma_2 = (P_2^T+Z_2^T)(\hat{F}_1\Phi_1+\hat{F}_2\Phi_2)+X_{20}.$$

5. Approximate the performance index

We can approximate the first term of Eq.(18) in terms of TF s as the following

$$\frac{1}{2}X^T(t_f)\,S\,X(t_f) \approx \frac{1}{2}\big[X_1^T\mathbf{T1}(T)\,S\,\mathbf{T1}^T(T)X_1 + X_1^T\mathbf{T1}(T)\,S\,\mathbf{T2}^T(T)X_2$$
$$+ X_2^T\mathbf{T2}(T)\,S\,\mathbf{T1}^T(T)X_1 + X_2^T\mathbf{T2}(T)\,S\,\mathbf{T2}^T(T)X_2\big]. \tag{25}$$

Without loss of generality, let $Q(t)$ and $R(t)$ be constant matrices and can be removed from the integrals, by using Eq. (12), we get

$$\frac{1}{2}\int_0^T X^T(t)Q(t)X(t)dt \approx \frac{h}{6}\left[X_1^TQX_1 + X_1^TQX_2 + X_2^TQX_2\right], \tag{26}$$

similarly, we have

$$\frac{1}{2}\int_0^T U^T(t)R(t)U(t)dt \approx \frac{h}{6}\left[U_1^TRU_1 + U_1^TRU_2 + U_2^TRU_2\right]. \tag{30}$$

6. Solution of the optimal control problem

Consider the following parameter optimization problem

$$J^*(X,U,\lambda) = J(X,U) + \lambda_1^T\omega_1^*(X_1,U_1) + \lambda_2^T\omega_2^*(X_2,U_2), \tag{31}$$

where λ is $2m-$vector represent the Lagrange multipliers.

The necessary conditions for the minimum are given by

$$\frac{\partial J^*}{\partial X_i}=0,$$
$$\frac{\partial J^*}{\partial U_i}=0,$$
$$\frac{\partial J^*}{\partial \lambda_i}=0, \quad i=1,2. \qquad (32)$$

There exists a system of $6m$ equations and $6m$ unknowns which can be solved for coefficients $\{x_i\}_{i=0}^{m}, \{u_i\}_{i=0}^{m}$ and $\{\lambda_i\}_{i=1}^{2m}$ in Eq. (32). Then the optimal control solution $u_m(t)$ and optimal trajectory $x_m(t)$ in Eq. (31) will be obtained.

7. Numerical example

Example 1. Consider the delay system

$$\begin{aligned}\dot{x}(t)&=x(t-1)+u(t), \quad 0\le t\le 2,\\ \varphi(t)&=1, \quad -1\le t\le 0,\end{aligned} \qquad (33)$$

which under above conditions minimizes the performance index

$$J=\frac{1}{2}\left[10^5 x^2(2)+\int_0^2 u^2(t)dt\right]. \qquad (34)$$

Using Eq. (14), it yields

$$\omega_1^*(X,U)=[I-P_1^T D_{T1}^T]X_1-P_1^T D_{T2}^T X_2-P_1^T U_1-P_1^T U_2-\gamma_1=0,$$
$$\omega_2^*(X,U)=-P_2^T D_{T1}^T X_1+[I-P_2^T D_{T2}^T]X_2-P_2^T U_1-P_2^T U_2-\gamma_2=0,$$

where

$$\gamma_1=X_{10}+(P_1^T+Z_1^T)(\Phi_1+\Phi_2),$$
$$\gamma_2=X_{20}+(P_2^T+Z_2^T)(\Phi_1+\Phi_2).$$

The Eq. (34) can be approximated by TF s as follows

$$J=\frac{1}{2}10^5\left[X_1^2T1(2)+X_2^2T2(2)\right]+\frac{h}{6}\left[U_1^TU_1+U_1^TU_2+U_2^TU_2\right].$$

Also, the value of performance index for $m=64$ is

$$J=1.837500.$$

By using the proposed approximation, the results for optimal control $u(t)$ are exact and the numerical results for optimal trajectory $x(t)$ are listed in Table 1.

t	$x_{TF}\ (m=64)$	method in [3]	method in [4]	x_{exact}
0.0	1.000000	1.000043	0.999532	1.000000
0.2	0.801123	0.800846	0.802412	0.801000
0.4	0.644082	0.644449	0.643181	0.644000
0.6	0.529082	0.528564	0.526945	0.529000
0.8	0.456123	0.456059	0.459392	0.456000
1.0	0.425000	0.424890	0.424715	0.425000
1.2	0.394296	0.394360	0.391268	0.394400

1.4	0.328146	0.328484	0.330558	0.328200
1.6	0.234764	0.234327	0.235159	0.234800
1.8	0.122569	0.122659	0.121131	0.122600
2.0	0.000006	0.000182	0.000544	0.000000

Table 1. Numerical results of state vector for Exa. 1.

8. Conclusion

In present work, a numerical method was developed to obtain the solution of optimal control of delay system. The proposed method is based on orthogonal triangular functions and their attractive properties. By using TF s, the solution of optimal control of time- delay systems can be reduced to the solution of algebraic equations. This method can be used to multi delay systems.

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