Chaos Synchronization of Fractional-Order Chaotic Lorenz-Stenflo System via Fractional Sliding Mode Control

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Abstract

In this paper, a fractional control strategy, PD^{λ} sliding mode control, is proposed to realize the synchronization of two chaotic Lorenz-Stenflo (LS) systems in master-slave configuration. According to the Lyapunov stability theory, the stability of the closed-loop error system is guaranteed. The control scheme is also robust to the system uncertainties and external disturbances. Chaos synchronization is obtained by proper choice of the control parameters. According our knowledge, this is the first research to apply fractional sliding mode controller for synchronization of LS system. The simulation results demonstrate the effectiveness of proposed control method.

Key words: Chaotic System, Synchronization, Fractional Sliding Mode Control, Lorenz-Stenflo.

1. Introduction

Fractional calculus is a 300-year-old mathematical topic [1,2]. Although it has so long a history, it did not attract much attention until the recent decades [3-5]. It was found that many systems in interdisciplinary fields can be described by the fractional differential equations, such as dielectric polarization, electrode-electrolyte polarization and electromagnetic waves [3,4]. Some fractional systems can also behave chaotically; for example, see the fractional Chua system [6], the fractional Lü system [7], and the fractional Chen system [8].

In 1996, Stenflo originally used a four-order autonomous chaotic system to describe the low-frequency short-wavelength gravity wave disturbance in the atmosphere [1]. This system is similar to the celebrated Lorenz equation but more complex than it due to the introduction of a new control parameter and a new state variable [2]. The complex dynamical behaviors (including chaos) of the Lorenz–Stenflo (LS) system have been investigated in [3–5].

Due to its potential applications in many domains, synchronization of chaotic systems is worth investigating and has attracted more and more attention. Chaos synchronization refers to dynamical synchrony of several chaotic systems through special coupling or by means of control. In recent years, study on the synchronization of fractional chaotic systems has attracted interest of many researchers. As a representative example, the LS system is not an exception too. It has been verified in theory and numerical simulation that two coupled Lorenz–Stenflo systems by some feedback controller can achieve chaos synchronization, e.g. by a linear state error feedback controller [6], by an active controller [2], and by an adaptive controller [7].

The problem of chaos control in fractional order systems has been addressed in the literatures [9-12], where controllers have been designed to eliminate the chaotic behaviour from system trajectories. In the researches, linear and nonlinear controllers have been proposed. Among the various methods developed to control fractional chaotic systems, the sliding mode control has been extensively utilized. It is attractive because it offers fast response, good transient response and it is also insensitive to uncertainly in the system. Usually, the designing process is divided into two steps. The first step is to design a switching surface and ensure that the sliding mode equation is stable. The second step is to design a controller that can drive the system states to the sliding surface.

This paper is devoted to apply the fractional sliding mode control for synchronization of two coupled Lorenz–Stenflo systems. According our knowledge, this is the first research to apply fractional sliding mode controller for synchronization of LS system. In this paper, a fractional control strategy, PD^{λ} sliding mode control, is proposed to realize the synchronization of two chaotic LS systems in master-slave configuration. According to the Lyapunov stability theory, the stability of the closed-loop error system is guaranteed. The control scheme is also robust to the system uncertainties and external disturbances.

The rest of this paper is organized as follows: In section 2, the basic concept of fractional calculus is presented. In section 3, the dynamic of fractional-order chaotic LS system is studied. In section 4 a fractional sliding surface is proposed to achieve robust synchronization between two perturbed LS systems. Moreover, numerical simulation results are presented in section 5 to show the effectiveness of the proposed method. Finally, the paper is concluded in section 6.

2. Fractional Calculus

Fractional calculus has three centuries of a history as the conventional calculus, but had not been very popular in the science and/or engineering communities. During past three centuries, the subject was with mathematicians, and only in the last few years, this was pulled to several (applied) fields of engineering, physical sciences and economics. Perhaps the fractional calculus will be the calculus of the twenty-first century. The fractional-order differentiator can be denoted by a general fundamental operator ${}_aD_t^q$ as a generalization of the differential and integral operators, which is defined as follows [13]:

$$ {}_aD_t^q = \begin{cases} \dfrac{d^q}{dt^q}, & R(q) > 0 \\ 1, & R(q) = 0 \\ \int_a^t (dt)^{-q}, & R(q) < 0 \end{cases} \tag{1} $$

where q is the fractional order commensurate which can be a complex number, however a constant q is assigned as initial condition. There are two commonly used definitions for the general fractional differentiation and integration, i.e., the Grünwald–Letnikov (GL) and the Riemann–Liouville (RL). The GL definition is:

$$ {}_aD_t^q f(t) = \lim_{h\to\infty} \frac{1}{h^q} \sum_{j=0}^{[(t-q)/h]} (-1)^j \binom{q}{j} f(t-jh) \tag{2}$$

where $[\cdot]$ is a flooring-operator, while the RL definition is given by:

$$ {}_aD_t^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q-n+1}} d\tau, \ (n-1<q<n) \tag{3}$$

where $\Gamma(x)$ is the well-known Euler's gamma function. The Caputo's definition can be written as

$$ L\left\{ {}_0D_t^\alpha \right\} = s^\alpha F(s), (n-1<q<n) \tag{4}$$

The initial conditions for the fractional order differential equations with the Caputo's derivatives are in the same form as for the integer-order differential equations. The formula for the Laplace transform of the RL fractional derivative has the form [15,16]:

$$ \int_0^\infty e_0^{st} D_t^\alpha dt = s^\alpha F(s) - \sum_{k=0}^{n-1} {}_0D_t^{\alpha-k-1} f(t)|_{t=0} \tag{5}$$

where s denotes the Laplace operator. For zero initial conditions, Laplace transform of fractional derivatives (Grunwald–Letnikov, Riemann–Liouville, and Caputo's) reduces to:

$$ L\left\{ {}_0D_t^\alpha \right\} = s^\alpha F(s) \tag{6}$$

3. System Description

The autonomous differential equations of LS system are described by:

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + dx_4 \\ \dot{x}_2 = x_1(c - x_3) - x_2 \\ \dot{x}_3 = x_1x_2 - bx_3 \\ \dot{x}_4 = -x_1 - ax_4 \end{cases} \tag{7}$$

where (a,b,c,d)=(1,0.7,26,1.5) [14,17]. The LS system is similar to the famous Lorenz system, but differs from it in the new control parameter d and the new state variable x_4. Change the integer-order LS system to the following fractional-order form:

$$\begin{cases} d^{q_1}x_1/dt^{q_1} = a(x_2 - x_1) + dx_4 \\ d^{q_2}x_2/dt^{q_2} = x_1(c - x_3) - x_2 \\ d^{q_3}x_3/dt^{q_3} = x_1x_2 - bx_3 \\ d^{q_4}x_4/dt^{q_4} = -x_1 - ax_4 \end{cases} \tag{8}$$

where $0 < q_i < 1$ (i = 1,2,3,4). The attractor and spectrum of 3.92 - order (q_i =0.98) LS system are shown in Figs. 1□ 3. This system has three equilibriums as follows:

$$ S_o = (0,0,0,0), S_\pm = \left(\pm p, \pm \frac{(a^2+d)}{a^2} p, \pm \frac{(a^2+d)}{a^2 b} p, \mp \frac{1}{a} p \right)$$

where $p^2 = (\frac{a^2bc}{a^2+d} - b)$, if p > 0. The Jacobian matrix of system evaluated at equilibrium point $(x_1^*, x_2^*, x_3^*, x_4^*)$ is obtained as $J = \begin{bmatrix} -1 & 1 & 0 & 1.5 \\ 26 - x_3^* & -1 & -x_1^* & 0 \\ x_2^* & x_1^* & -0.7 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}$

By considering $(a,b,c,d) = (1, 0.7, 26, 1.5)$, the equilibrium point and their corresponding eigenvalues are respectively: $S_o = (0,0,0,0)$, $S_{\pm} = \left(\pm 2.5652, \pm 6.4130, \pm 23.5, \mp 2.5652\right)$, $\lambda_o = (3.9497, -5.9497, -1, -0.7)$, $\lambda_+ = \left(0.1771 + j\,2.8243, 0.1771 - j\,2.8243, -2.0543, -2\right)$, and $\lambda_- = (-7.6231, 5.1648, 0, -1.2417)$.

Suppose λ is an unstable eigenvalue of one of the saddle points of index 2. A necessary condition for fractional system $D^q x = f(x)$ to remain chaotic is keeping the eigenvalue λ in the unstable region. This means

$$\tan(q\pi/2) > \frac{|\mathrm{Im}(\lambda)|}{\mathrm{Re}(\lambda)} \quad \Rightarrow \quad q > \frac{2}{\pi}\tan^{-1}\left(\frac{|\mathrm{Im}(\lambda)|}{\mathrm{Re}(\lambda)}\right) \tag{9}$$

According to Eq. 9, maximum fractional-order q that for which the LS system with commensurate fractional-order and above parameters can demonstrate chaos is $q \approx 0.96$. Fig. 4 shows stable and unstable regions in this case.

4. Synchronization Method

In this section, the synchronization method is studied o achieve complete synchronization for fractional-order LS system. By considering system defined in Eq. 7 as the master system, the corresponding response (slave) is described by:

$$\begin{cases} d^{q_1} y_1 / dt^{q_1} = a(x_2 - x_1) + dx_4 + \Delta f(y_i) + d(t) + u_1 \\ d^{q_2} y_2 / dt^{q_2} = x_1(c - x_3) - x_2 + u_2 \\ d^{q_3} y_3 / dt^{q_3} = x_1 x_2 - b x_3 \\ d^{q_4} y_4 / dt^{q_4} = -x_1 - a x_4 \end{cases} \tag{10}$$

Define the complete synchronization errors by $e_i = y_i - x_i \; (i = 1, \ldots, 4)$. Thus the error system has following structures:

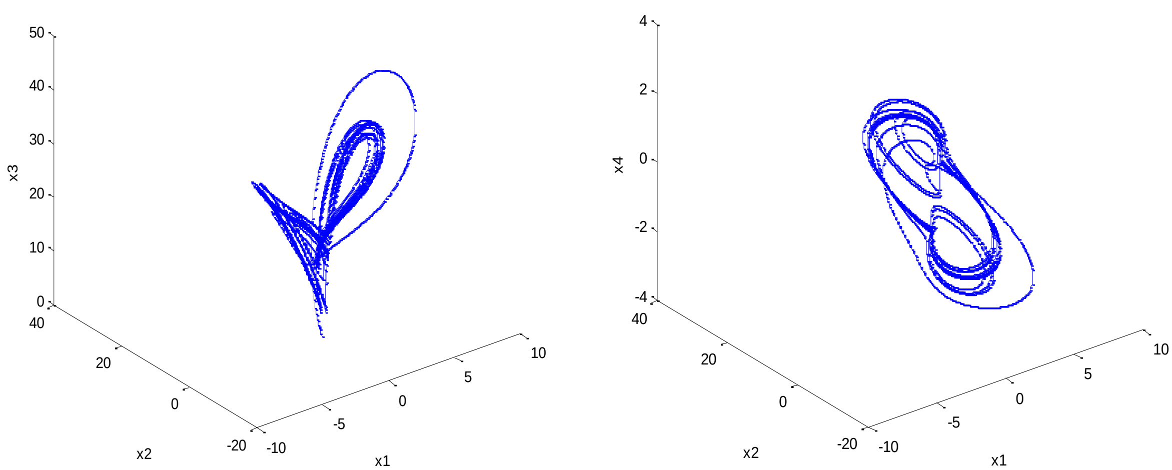


Fig 1: The attractor of 3.92-order LS system in $(x_2 - x_3$

Fig 2: The attractor of 3.92-order LS system in

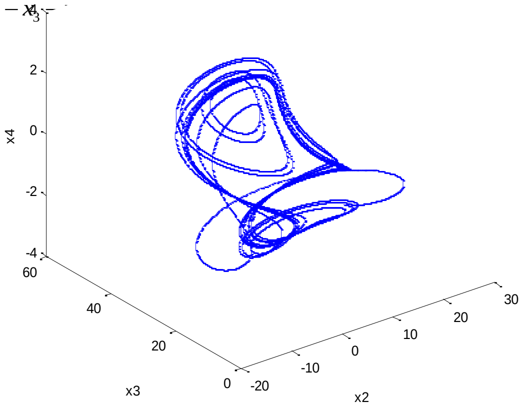


Fig 3: The attractor of 3.92-order LS system in $(x_2 - x_3 - x_4)$

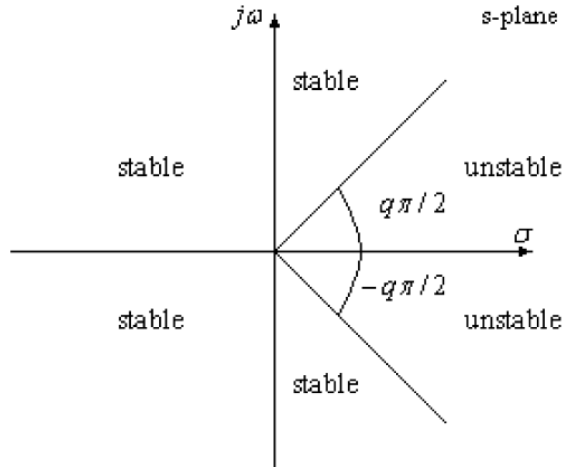


Fig 4: Stability region of linear fractional-order system with order q.

$$\begin{cases} d^{q_1}e_1/dt^{q_1} = a(e_2 - e_1) + de_4 + \Delta f(y_i) + d(t) + u_1 \\ d^{q_2}e_2/dt^{q_2} = ce_1 - e_2 - y_1y_3 + x_1x_3 + u_2 \\ d^{q_3}e_3/dt^{q_3} = be_3 + y_1y_2 - x_1x_2 \\ d^{q_4}e_4/dt^{q_4} = -e_1 - ae_4 \end{cases} \tag{11}$$

where $\Delta f = 0.5 - \sin(\pi y_1)\sin(2\pi y_2)\sin(3\pi y_3)$ is uncertainty whereas $d(t) = 0.2\cos(\pi t)$ is disturbance. Initial conditions for master and slave systems are chosen as follows:
$(x_1(0), x_2(0), x_3(0), x_4(0)) = (0.1, 0.2, 0.2, -0.2), (y_1(0), y_2(0), y_3(0), y_4(0)) = (-0.2, 0.2, -0.2, 0.1)$
The fractional PD switching surface are [18,19]:

$$\begin{cases} S_1 = K_{p_1}e_1 + K_{d_1}D^{\lambda}e_1 \\ S_2 = K_{p_2}e_2 + K_{d_2}D^{\lambda}e_2 \end{cases} \tag{12}$$

where $K_{pi}, K_{di} > 0$ and S_i, $i = 1,2$ is called the sliding surface. The control objective can be achieved by choosing the control input such that the sliding surface satisfies the following sufficient condition:

$$\frac{1}{2}\frac{d}{dt}S_i^2 \le -\eta_i |S_i| \tag{13}$$

where η is a positive constant. Once the behaviour of system is settled on the surface, it is called the sliding mode $(\dot{S} = 0)$ has occurred. Taking the time derivative on both sides of Eq. 12, we can obtain

$$\begin{cases} \dot{S}_1 = K_{p_1}\dot{e}_1 + K_{d_1}D^{1+\lambda-q}(D^q e_1) \\ \dot{S}_2 = K_{p_2}\dot{e}_2 + K_{d_2}D^{1+\lambda-q}(D^q e_2) \end{cases} \tag{14}$$

Using Eq. 11 and then forcing $\dot{S}_i = 0$, the input control signal results as

$$\begin{cases} u_1 = -\dfrac{K_{p_1}}{K_{d_1}} D^{q-\lambda} e_1 - a(e_2 - e_1) - de_4 - K_{s_1} sat(S_1 / \varphi_1) \\ u_2 = -\dfrac{K_{p_2}}{K_{d_2}} D^{q-\lambda} e_2 - ce_1 + e_2 + y_1 y_3 - x_1 x_3 - K_{s_2} sat(S_2 / \varphi_2) \end{cases} \tag{15}$$

where

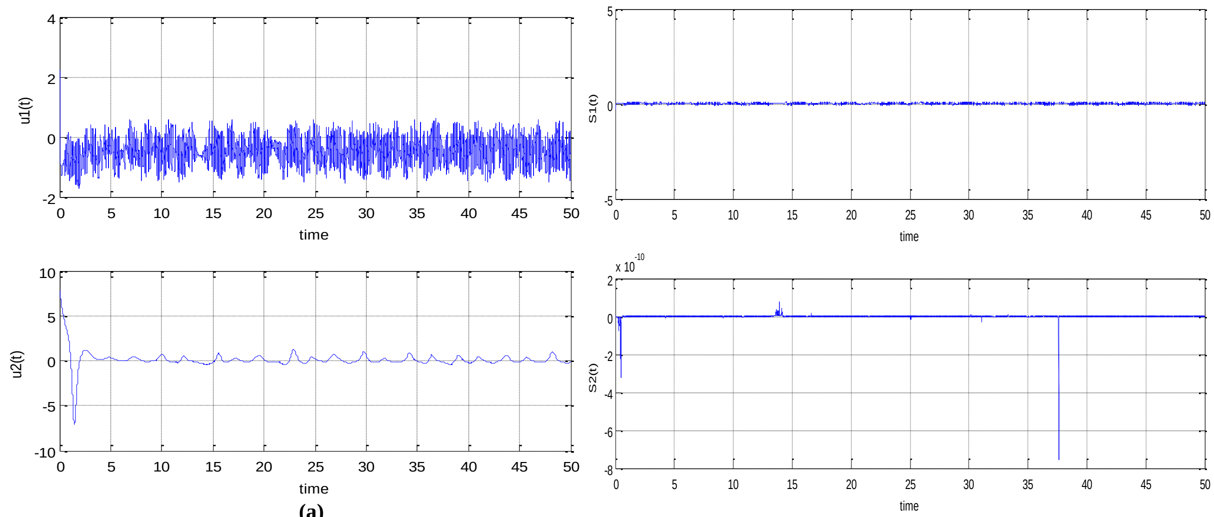
$$sat\left(\frac{S}{\varphi}\right) = \begin{cases} 1, & if\ \left(\dfrac{S}{\varphi}\right) \geq 1 \\ \dfrac{S}{\varphi}, & if\ -1 < S < 1 \\ -1, & if\ \left(\dfrac{S}{\varphi}\right) \leq -1 \end{cases} \tag{16}$$

5. Simulation Results

The parameter values in simulations are chosen as $K_{pi} = 1,\ K_{di} = 1,\ K_{si} = 10,\ q = 0.98,\ \lambda = 0.97$ and $\varphi = 1$. To evaluate the robustness of the control scheme, a system uncertainty term $\Delta f = 0.5 - \sin(\pi y_1)\sin(2\pi y_2)\sin(3\pi y_3)$ is added. Moreover, the external disturbance d is injected of the system as $d(t) = 0.2\cos(\pi t)$. Simulation results are shown in Figs. 5□6. Fig. 5(a) shows that control signals u with saturation function whereas Fig. 5(b) shows that surface functions S for master and slave system. Fig. 6(a) depicts that time response of controlled chaotic LS synchronization system. Moreover, Fig. 6(b) represents that synchronization error. The simulation results demonstrate the effectiveness of proposed control method in presence of uncertainty and disturbance.

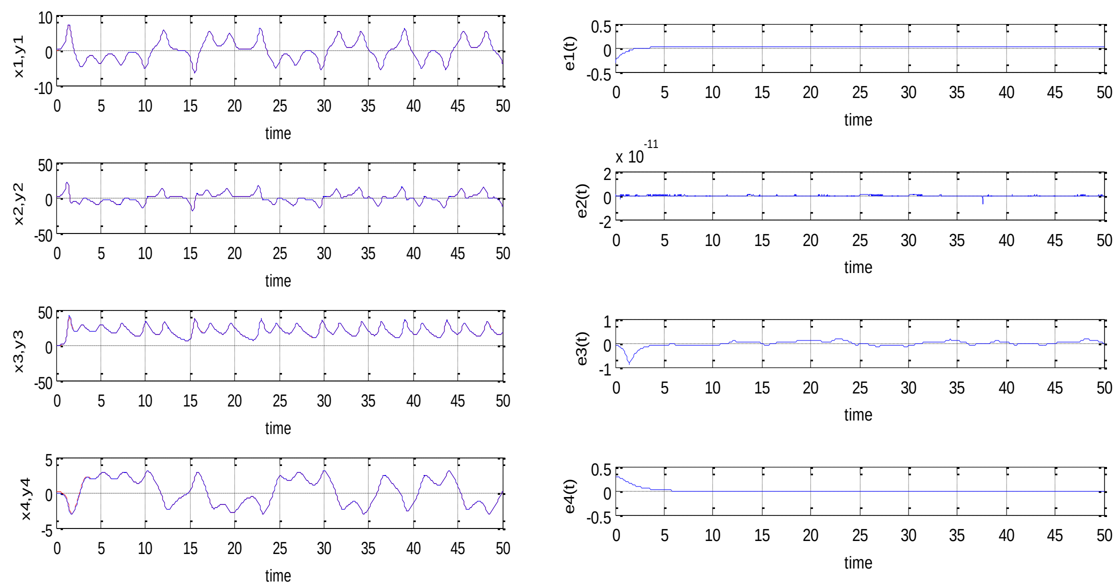
6. Conclusion

Keeping the control under uncertainty conditions is one of the most complicated control objectives. In this paper synchronization of fractional type order of chaotic Lorenz-Stenflo (LS) system with uncertainties and disturbance have been investigated. A robust fractional sliding control scheme was proposed to achieve the synchronization problem for fractional type order of chaotic Lorenz-Stenflo (LS) system in a master-slave structure with uncertainty and external disturbance was controlled by a fractional controller. The fractional sliding mode controller schemes are less sensitive to the parameter variations (uncertainties and disturbance). It was therefore concluded that the robustness and lesser sensitivity to disturbances. The numerical simulations were included to illustrate the feasibility and effectiveness of the proposed control scheme.



(a) (b)

Fig 5: **(a)** control signals u ,with saturation function, **(b)** surface functions S for master and slave system.



(a) (b)

Fig 6: **(a)** Time response of controlled chaotic LS synchronization system, **(b)** Synchronization error.

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