Numerical Solution of System of Volterra integro differential equations

H. Almasieh
Department of Mathematics, Islamic Azad University, Khorasgan (Isfahan) Branch, Isfahan, Iran
Email: H.Almasieh@khuisf.ac.ir
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Name of the Presenter: H. Almasieh

Abstract

A numerical method based on orthogonal triangular functions is developed for solving the system of Volterra integro-differential equations. By using triangular functions and their properties in obtaining operational matrices of product and integration, the solution of Volterra integro-differential equations system can be reduced to the solution of algebraic equations. The proposed method can be applied to obtain the solution of Fredholm integro-differential equations system.

Key words: Triangular Functions, Integro-Differential Equations System, Operational Matrices.

1. Introduction

Integro-differential equations (IDE)s arise in many branches of science and engineering, for example in control theory, financial mathematics and chemistry. Also, the physical systems are modeled under the differential sense, it finally gives a differential equation, an integral equation or an integro-differential equation. Forthcoming of the first two equations mostly appear in the last equation. There are many numerical methods to solve (IDE)s e.g. interpolation and Chebyshev polynomials technique [7], hybrid functions [6], Rationalized Haar functions method [5], Homotopy perturbation method [2] and etc. our focus in present paper is on the system of Volterra type IDE s. Let us first review of orthogonal triangular functions (TF)s .

2.Review of orthogonal triangular functions (TF)s

Definition 1. A set of BPF , $\mathbf{\Psi}(t)$ containing m component functions in the semi-open interval $[0,T)$ is given by

$$\mathbf{\Psi(t)} = \left[\psi_0(t) \quad \psi_1(t) \quad \dots \quad \psi_i(t) \quad \dots \quad \psi_{m-1}(t)\right]^T, \tag{1}$$

where $[...]^T$ denotes transpose.

The i th component $\psi_i(t)$ of the BPF vector $\mathbf{\Psi}(t)$ is defined as

$$\psi_i(t) = \begin{cases} 1 \quad , & \dfrac{iT}{m} \le t < \dfrac{(i+1)T}{m}, \\ 0 \quad , & otherwise, \end{cases} \tag{2}$$

where $i = 0,1,2,...,(m-1)$ and $h = \dfrac{T}{m}$, [4].

2.1 *BPF* expansion of functions

A function $x(t) \in L^2[0,T)$ can be expanded in terms of BPF s as follows

$$x(t) \approx \left[x_0 \quad x_1 \quad x_2 \quad \dots \quad x_i \quad \dots \quad x_{m-1}\right] \mathbf{\Psi(t)} = X^T \mathbf{\Psi}(t), \tag{3}$$

where the constant coefficients x_i's are given by

$$x_i = (\frac{1}{h})\int_{ih}^{(i+1)h} x(t)\, dt. \tag{4}$$

Definition 2. Let $\psi_i(t)$ be the i th member of an m-set BPF, we have dissected a BPF into two TF s as

$$\psi_i(t) = T1_i(t) + T2_i(t), \tag{5}$$

where $T1_i(t)$ and $T2_i(t)$ are the i th components of the $m-$vectors $T1(t)$ and $T2(t)$, which are defined in [4] as follows**:**

$$T1_i(t) = \begin{cases} 1 - \dfrac{(t-ih)}{h}, & ih \le t < (i+1)h \\ 0, & otherwise, \end{cases} \tag{6}$$

and

$$T2_i(t) = \begin{cases} \dfrac{(t-ih)}{h}, & ih \le t < (i+1)h \\ 0, & otherwise, \end{cases} \tag{7}$$

for $i = 0,1,2,...,(m-1)$.

2.2 *TF* expansion of functions

(**i**) Suppose $f(t)$ be an $L^2[0,T)$, the expansion of f in terms of TF s is as the following

$$\begin{aligned} f(t) &\approx \sum_{i=0}^{m-1} \left[f_i T1_i(t) + f_{i+1} T2_i(t)\right] \\ &= \left[f_0 \quad f_1 \quad \dots \quad f_i \quad \dots \quad f_{m-1}\right] \mathbf{T1}(t) + \left[f_1 \quad f_2 \quad \dots \quad f_i \quad \dots \quad f_m\right] \mathbf{T2}(t) \\ &= F_1^T\, \mathbf{T1}(t) + F_2^T\, \mathbf{T2}(t). \end{aligned} \tag{8}$$

where the sequence of coefficients $\{f_i\}_{i=0}^{m}$ are given by

$$f_i = f(ih), \quad i = 0,1,...,m. \tag{9}$$

(**ii**) Suppose $g(t)$ be an L^2 function on $[0,T)$, by expanding $f(t)$ and $g(t)$ in terms of TF s, we get

$$f(t)g(t) = F_1^T \hat{G}_1\, \mathbf{T1}(t) + F_2^T \hat{G}_2\, \mathbf{T2}(t), \tag{10}$$

where $\hat{G}_1$ and $\hat{G}_2$ are diagonal $m \times m$ matrices and we have

$$\hat{G}_1 = \begin{pmatrix} g_0 & 0 & \dots & 0 \\ 0 & g_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_{m-1} \end{pmatrix}, \qquad \hat{G}_2 = \begin{pmatrix} g_1 & 0 & \dots & 0 \\ 0 & g_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_m \end{pmatrix}. \tag{11}$$

(**iii**) Also, each function $f(t,s) \in L^2([0,T) \times [0,T))$, can be expanded with respect to TF s as the following

$$f(t,s) \approx \mathbf{T1}^T(t)F11\mathbf{T1}(s) + \mathbf{T1}^T(t)F12\mathbf{T2}(s) + \mathbf{T2}^T(t)F21\mathbf{T1}(s) + \mathbf{T2}^T(t)F22\mathbf{T2}(s), \tag{12}$$

where $F11, F12, F21$ and $F22$ are $m \times m$ matrices which can be obtained as [1],

$$\begin{aligned} (F11)_{ij} &= f(ih, jh), \\ (F12)_{ij} &= f(ih, (j+1)h), \\ (F21)_{ij} &= f((i+1)h, jh), \\ (F22)_{ij} &= f((i+1)h, (j+1)h), \quad i, j = 0, \dots, m-1. \end{aligned} \tag{13}$$

(**iv**) Similarly, suppose $f(t,s) \in L^2([0,T) \times [0,T))$ and $g(s) \in L^2[0,T)$, by expanding these functions in terms of TF s, we get

$$\begin{aligned} f(t,s)g(s) \approx{}& \mathbf{T1}^T(t)F11\hat{G}_1\mathbf{T1}(s) + \mathbf{T1}^T(t)F12\hat{G}_2\mathbf{T2}(s) \\ &+ \mathbf{T2}^T(t)F21\hat{G}_1\mathbf{T1}(s) + \mathbf{T2}^T(t)F22\hat{G}2\mathbf{T2}(s), \end{aligned} \tag{14}$$

where $\hat{G}_1$ and $\hat{G}_2$ are diagonal matrices which can be obtained as in Eq. (11).

2.3 Operational matrices of integration

The operational matrices of integration for TF s have been given by Deb in [4], as follows

$$\int_0^t \mathbf{T1}(\tau)\,d\tau = \int_0^t \mathbf{T2}(\tau)\,d\tau = P_1\mathbf{T1}(t) + P_2\mathbf{T2}(t), \tag{15}$$

where P_1 and P_2 are $m \times m$ matrices as follows

$$P_1 = \frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad P_2 = \frac{h}{2}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{16}$$

3. Problem statement

Consider a system of Volterra integro-differential equations as the following

$$\sum_{j=1}^{n}\left[\left(r_{ij}(s)y'_j(s) + q_{ij}(s)y_j(s)\right) + \int_0^s k_{ij}(s,t)y'_j(t)dt + \int_0^s h_{ij}(s,t)y_j(t)dt\right] = x_i(s), \tag{17}$$

for $i = 1,2,\dots,n,$ with the initial condition

$$y_i(0) = y_{i0}, \quad i = 1,2,\dots,n,$$

where $x_i(s), r_{ij}(s), q_{ij}(s) \in L^2[0,T)$ and $k_{ij}(s,t), h_{ij}(s,t) \in L^2([0,T)\times[0,T))$, for $i=1,...,n$ are known functions and $y(t)=[y_1(t), y_2(t),...y_n(t)]^T$ are unknown functions.

3.1 Approximate the solution of system of IDEs

By using Eq. (10) in Eq. (17), we have

$$r_{ij}(s)y'_j(s) \approx Y_{1j}'^T \hat{R}_{1ij}\,\mathbf{T1}(s) + Y_{2j}'^T \hat{R}_{2ij}\,\mathbf{T2}(s), \tag{18}$$

and

$$q_{ij}(s)y_j(s) \approx Y_{1j}^T \hat{Q}_{1ij}\,\mathbf{T1}(s) + Y_{2j}^T \hat{Q}_{2ij}\,\mathbf{T2}(s), \tag{19}$$

where $\hat{R}_{1ij}, \hat{R}_{2ij}, \hat{Q}_{1ij}$ and $\hat{Q}_{2ij}$ are diagonal matrices such that their diagonal elements are the elements of R_{1ij}, R_{2ij}, Q_{1ij}, and Q_{2ij}, respectively.

Similarly, by using Eqs. (14) and (15), we get

$$\int_0^s k_{ij}(s,t)y'_j(t)dt \approx \tilde{Z}_{1j}^T\,\mathbf{T1}(s) + \tilde{Z}_{2j}^T\,\mathbf{T2}(s), \tag{20}$$

where

$$\tilde{Z}_{1j} = P_1^T\left(\hat{Y}_{1j}\,K11_{ij}^T + \hat{Y}_{2j}\,K12_{ij}^T\right),$$
$$\tilde{Z}_{2j} = P_2^T\left(\hat{Y}_{1j}\,K21_{ij}^T + \hat{Y}_{2j}\,K22_{ij}^T\right).$$

We know that

$$y(s) = \int_0^s y'(\tau)\,d\tau + y(0), \tag{21}$$

Using the expansion of TFs in Eq. (21), applying Eq. (15) and equating the like coefficients of TFs, They yield

$$Y_{1j} \approx Y_{01} + P_1^T Y'_{1j} + P_1^T Y'_{2j},$$
$$Y_{2j} \approx Y_{02} + P_2^T Y'_{1j} + P_2^T Y'_{2j}. \tag{22}$$

By using Eqs. (18)~(22) in Eq. (17) and expanding $x_i(s)$ in terms of TF, there exists a system of $6mn$ equations and $6mn$ unknowns which can be solved for coefficients $\{Y_{1j}\}_{j=1}^n$ and $\{Y_{2j}\}_{j=1}^n$.

4. Numerical Example

In order to show the numerical efficiency and simplicity of proposed method, the numerical results of an example are provided by $MATLAB$ software. The errors are obtained as the following

$$e(y(s)) = \left|\,\bar{y}(s) - y_m(s)\right|,$$

where $\bar{y}(s)$ and $y_m(s)$ are exact and approximate solutions, respectively.

Example 1. Consider the system of linear Volterra integro- differential equations [3],

$$y_1'(s)=1+s+s^2-y_2(s)-\int_0^s\left[y_1(t)+y_2(t)\right]dt,$$

$$y_2'(s)=-1-s+y_1(s)-\int_0^s\left[y_1(t)-y_2(t)\right]dt,$$

with the initial conditions

$$y_1(0)=1,\quad y_2(0)=-1.$$

The exact solution is $y(s)=\left[y_1(s),\,y_2(s)\right]^T=\left[s+e^s,s-e^s\right]^T$.

A comparison of TF method with $m=500$ and the ADM, [3] and HPM, [8] methods are obtained in Table 1. The errors compared with ADM and HPM are smaller when t increases. Thus, in TF method, the errors are stable as t increases but for the ADM and HPM, the errors increases as t increases.

t	$e(y_{TF})$	$e(y_{ADM})$	$e(y_{HPM})$
0.0	(0,0)	(0,0)	(0,0)
0.1	(0.4E-7, 0.4E-7)	---------	---------
0.2	(0.8E-7, 0.8E-7)	(0.3E-8, 0.3E-8)	(0.3E-8, 0.3E-8)
0.3	(0.1E-6, 0.1E-6)	---------	---------
0.4	(0.2E-6, 0.2E-6)	(0.3E-6, 0.3E-6)	(0.3E-6, 0.3E-6)
0.5	(0.3E-6, 0.3E-6)	---------	---------
0.6	(0.4E-6, 0.4E-6)	(0.5E-5, 0.5E-5)	(0.5E-5, 0.5E-5)
0.7	(0.5E-6, 0.5E-6)	---------	---------
0.8	(0.6E-6, 0.6E-6)	(0.4E-4, 0.4E-4)	(0.4E-4, 0.4E-4)
0.9	(0.7E-6, 0.7E-6)	---------	---------

Table 1. Comparison of TF method with ADM and HPM with m=500.

5. Conclusion

In present work, a numerical method was depeloped to obtain the solution of Volterra integro-differential equations system. The proposed method is based on orthogonal triangular functions and their attractive properties. By using TFs, the solution of system of Volterra integro-differential equations can be reduced to the solution of algebraic equations. This method can be used to system of integral equations.

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