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Numerical Solution of System of Volterra integro differential equations



H. Almasieh

Department of Mathematics, Islamic Azad University,
Khorasgan (Isfahan) Branch, Isfahan, Iran

Email: H.Almasieh@khuisf.ac.ir

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Abstract

A numerical method based on orthogonal triangular functions is developed for solving the system of Volterra integro-differential equations. By using triangular functions and their properties in obtaining operational matrices of product and integration, the solution of Volterra integro-differential equations system can be reduced to the solution of algebraic equations. The proposed method can be applied to obtain the solution of Fredholm integro-differential equations system.

Key words: Triangular Functions, Integro-Differential Equations System, Operational Matrices.

1. Introduction

Integro-differential equations (*IDE*)s arise in many branches of science and engineering, for example in control theory, financial mathematics and chemistry. Also, the physical systems are modeled under the differential sense, it finally gives a differential equation, an integral equation or an integro-differential equation. Forthcoming of the first two equations mostly appear in the last equation. There are many numerical methods to solve (*IDE*)s e.g. interpolation and Chebyshev polynomials technique [7], hybrid functions [6], Rationalized Haar functions method [5], Homotopy perturbation method [2] and etc. our focus in present paper is on the system of Volterra type *IDE*s. Let us first review of orthogonal triangular functions (*TF*)s .

2. Review of orthogonal triangular functions (*TF*)s

Definition 1. A set of *BPF* , $\Psi(t)$ containing m component functions in the semi-open interval $[0, T)$ is given by

$$\Psi(\mathbf{t}) = [\psi_0(t) \ \psi_1(t) \ \dots \ \psi_i(t) \ \dots \ \psi_{m-1}(t)]^T, \quad (1)$$

where $[\dots]^T$ denotes transpose.

The i th component $\psi_i(t)$ of the *BPF* vector $\Psi(t)$ is defined as

$$\psi_i(t) = \begin{cases} 1 & , \quad \frac{iT}{m} \leq t < \frac{(i+1)T}{m}, \\ 0 & , \quad \textit{otherwise}, \end{cases} \quad (2)$$

where $i = 0, 1, 2, \dots, (m-1)$ and $h = \frac{T}{m}$, [4].

2.1 BPF expansion of functions

A function $x(t) \in L^2[0, T]$ can be expanded in terms of *BPF* s as follows

$$x(t) \approx [x_0 \ x_1 \ x_2 \ \dots \ x_i \ \dots \ x_{m-1}] \Psi(\mathbf{t}) = X^T \Psi(t), \quad (3)$$

where the constant coefficients x_i 's are given by

$$x_i = \left(\frac{1}{h}\right) \int_{ih}^{(i+1)h} x(t) dt. \quad (4)$$

Definition 2. Let $\psi_i(t)$ be the i th member of an m -set *BPF*, we have dissected a *BPF* into two *TF* s as

$$\psi_i(t) = T1_i(t) + T2_i(t), \quad (5)$$

where $T1_i(t)$ and $T2_i(t)$ are the i th components of the m -vectors $T1(t)$ and $T2(t)$, which are defined in [4] as follows:

$$T1_i(t) = \begin{cases} 1 - \frac{(t-ih)}{h}, & ih \leq t < (i+1)h \\ 0, & \textit{otherwise}, \end{cases} \quad (6)$$

and

$$T2_i(t) = \begin{cases} \frac{(t-ih)}{h}, & ih \leq t < (i+1)h \\ 0, & \textit{otherwise}, \end{cases} \quad (7)$$

for $i = 0, 1, 2, \dots, (m-1)$.

2.2 TF expansion of functions

(i) Suppose $f(t)$ be an $L^2[0, T]$, the expansion of f in terms of *TF* s is as the following

$$\begin{aligned} f(t) &\approx \sum_{i=0}^{m-1} [f_i T1_i(t) + f_{i+1} T2_i(t)] \\ &= [f_0 \ f_1 \ \dots \ f_i \ \dots \ f_{m-1}] \mathbf{T1}(t) + [f_1 \ f_2 \ \dots \ f_i \ \dots \ f_m] \mathbf{T2}(t) \\ &= F_1^T \mathbf{T1}(t) + F_2^T \mathbf{T2}(t). \end{aligned} \quad (8)$$

where the sequence of coefficients $\{f_i\}_{i=0}^m$ are given by

$$f_i = f(ih), \quad i = 0, 1, \dots, m. \quad (9)$$

(ii) Suppose $g(t)$ be an L^2 function on $[0, T]$, by expanding $f(t)$ and $g(t)$ in terms of *TF* s, we get

$$f(t)g(t) = F_1^T \hat{G}_1 \mathbf{T1}(t) + F_2^T \hat{G}_2 \mathbf{T2}(t), \quad (10)$$

where \hat{G}_1 and \hat{G}_2 are diagonal $m \times m$ matrices and we have

$$\hat{G}_1 = \begin{pmatrix} g_0 & 0 & \dots & 0 \\ 0 & g_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_{m-1} \end{pmatrix}, \quad \hat{G}_2 = \begin{pmatrix} g_1 & 0 & \dots & 0 \\ 0 & g_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_m \end{pmatrix}. \quad (11)$$

(iii) Also, each function $f(t, s) \in L^2([0, T] \times [0, T])$, can be expanded with respect to TF s as the following

$$f(t, s) \approx \mathbf{T1}^T(t)F11\mathbf{T1}(s) + \mathbf{T1}^T(t)F12\mathbf{T2}(s) + \mathbf{T2}^T(t)F21\mathbf{T1}(s) + \mathbf{T2}^T(t)F22\mathbf{T2}(s), \quad (12)$$

where $F11, F12, F21$ and $F22$ are $m \times m$ matrices which can be obtained as [1],

$$\begin{aligned} (F11)_{ij} &= f(ih, jh), \\ (F12)_{ij} &= f(ih, (j+1)h), \\ (F21)_{ij} &= f((i+1)h, jh), \\ (F22)_{ij} &= f((i+1)h, (j+1)h), \quad i, j = 0, \dots, m-1. \end{aligned} \quad (13)$$

(iv) Similarly, suppose $f(t, s) \in L^2([0, T] \times [0, T])$ and $g(s) \in L^2[0, T]$, by expanding these functions in terms of TF s, we get

$$\begin{aligned} f(t, s)g(s) &\approx \mathbf{T1}^T(t)F11\hat{G}_1\mathbf{T1}(s) + \mathbf{T1}^T(t)F12\hat{G}_2\mathbf{T2}(s) \\ &\quad + \mathbf{T2}^T(t)F21\hat{G}_1\mathbf{T1}(s) + \mathbf{T2}^T(t)F22\hat{G}_2\mathbf{T2}(s), \end{aligned} \quad (14)$$

where \hat{G}_1 and \hat{G}_2 are diagonal matrices which can be obtained as in Eq. (11).

2.3 Operational matrices of integration

The operational matrices of integration for TF s have been given by Deb in [4], as follows

$$\int_0^t \mathbf{T1}(\tau) d\tau = \int_0^t \mathbf{T2}(\tau) d\tau = P_1\mathbf{T1}(t) + P_2\mathbf{T2}(t), \quad (15)$$

where P_1 and P_2 are $m \times m$ matrices as follows

$$P_1 = \frac{h}{2} \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad P_2 = \frac{h}{2} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \quad (16)$$

3. Problem statement

Consider a system of Volterra integro-differential equations as the following

$$\sum_{j=1}^n \left[r_{ij}(s)y_j'(s) + q_{ij}(s)y_j(s) + \int_0^s k_{ij}(s,t)y_j'(t)dt + \int_0^s h_{ij}(s,t)y_j(t)dt \right] = x_i(s), \quad (17)$$

for $i = 1, 2, \dots, n$, with the initial condition

$$y_i(0) = y_{i0}, \quad i = 1, 2, \dots, n,$$

where $x_i(s), r_{ij}(s), q_{ij}(s) \in L^2[0, T]$ and $k_{ij}(s, t), h_{ij}(s, t) \in L^2([0, T] \times [0, T])$, for $i = 1, \dots, n$ are known functions and $y(t) = [y_1(t), y_2(t), \dots, y_n(t)]^T$ are unknown functions.

3.1 Approximate the solution of system of IDE s

By using Eq. (10) in Eq. (17), we have

$$r_{ij}(s)y'_j(s) \approx Y_{1j} {}^{tT} \hat{R}_{1ij} \mathbf{T1}(s) + Y_{2j} {}^{tT} \hat{R}_{2ij} \mathbf{T2}(s), \quad (18)$$

and

$$q_{ij}(s)y_j(s) \approx Y_{1j} {}^T \hat{Q}_{1ij} \mathbf{T1}(s) + Y_{2j} {}^T \hat{Q}_{2ij} \mathbf{T2}(s), \quad (19)$$

where $\hat{R}_{1ij}, \hat{R}_{2ij}, \hat{Q}_{1ij}$ and \hat{Q}_{2ij} are diagonal matrices such that their diagonal elements are the elements of $R_{1ij}, R_{2ij}, Q_{1ij}$, and Q_{2ij} , respectively.

Similarly, by using Eqs. (14) and (15), we get

$$\int_0^s k_{ij}(s, t)y'_j(t)dt \approx \tilde{Z}_{1j}^T \mathbf{T1}(s) + \tilde{Z}_{2j}^T \mathbf{T2}(s), \quad (20)$$

where

$$\begin{aligned} \tilde{Z}_{1j} &= P_1^T \left(\hat{Y}_{1j} K11_{ij}^T + \hat{Y}_{2j} K12_{ij}^T \right), \\ \tilde{Z}_{2j} &= P_2^T \left(\hat{Y}_{1j} K21_{ij}^T + \hat{Y}_{2j} K22_{ij}^T \right) \end{aligned}$$

We know that

$$y(s) = \int_0^s y'(\tau) d\tau + y(0), \quad (21)$$

Using the expansion of *TF* s in Eq. (21), applying Eq. (15) and equating the like coefficients of *TF* s, They yield

$$\begin{aligned} Y_{1j} &\approx Y_{01} + P_1^T Y'_{1j} + P_1^T Y'_{2j}, \\ Y_{2j} &\approx Y_{02} + P_2^T Y'_{1j} + P_2^T Y'_{2j}. \end{aligned} \quad (22)$$

By using Eqs. (18)~(22) in Eq. (17) and expanding $x_i(s)$ in terms of *TF*, there exists a system of $6mn$ equations and $6mn$ unknowns which can be solved for coefficients $\{Y_{1j}\}_{j=1}^n$ and $\{Y_{2j}\}_{j=1}^n$.

4. Numerical Example

In order to show the numerical efficiency and simplicity of proposed method, the numerical results of an example are provided by *MATLAB* software. The errors are obtained as the following

$$e(y(s)) = |\bar{y}(s) - y_m(s)|,$$

where $\bar{y}(s)$ and $y_m(s)$ are exact and approximate solutions, respectively.

Example 1. Consider the system of linear Volterra integro- differential equations [3],

$$y_1'(s) = 1 + s + s^2 - y_2(s) - \int_0^s [y_1(t) + y_2(t)]dt,$$

$$y_2'(s) = -1 - s + y_1(s) - \int_0^s [y_1(t) - y_2(t)]dt,$$

with the initial conditions

$$y_1(0) = 1, \quad y_2(0) = -1.$$

The exact solution is $y(s) = [y_1(s), y_2(s)]^T = [s + e^s, s - e^s]^T$.

A comparison of *TF* method with $m = 500$ and the *ADM*, [3] and *HPM*, [8] methods are obtained in Table 1. The errors compared with *ADM* and *HPM* are smaller when t increases. Thus, in *TF* method, the errors are stable as t increases but for the *ADM* and *HPM*, the errors increases as t increases.

t	$e(y_{TF})$	$e(y_{ADM})$	$e(y_{HPM})$
0.0	(0,0)	(0,0)	(0,0)
0.1	(0.4E-7, 0.4E-7)	-----	-----
0.2	(0.8E-7, 0.8E-7)	(0.3E-8, 0.3E-8)	(0.3E-8, 0.3E-8)
0.3	(0.1E-6, 0.1E-6)	-----	-----
0.4	(0.2E-6, 0.2E-6)	(0.3E-6, 0.3E-6)	(0.3E-6, 0.3E-6)
0.5	(0.3E-6, 0.3E-6)	-----	-----
0.6	(0.4E-6, 0.4E-6)	(0.5E-5, 0.5E-5)	(0.5E-5, 0.5E-5)
0.7	(0.5E-6, 0.5E-6)	-----	-----
0.8	(0.6E-6, 0.6E-6)	(0.4E-4, 0.4E-4)	(0.4E-4, 0.4E-4)
0.9	(0.7E-6, 0.7E-6)	-----	-----

Table 1. Comparison of TF method with ADM and HPM with $m=500$.

5. Conclusion

In present work, a numerical method was developed to obtain the solution of Volterra integro-differential equations system. The proposed method is based on orthogonal triangular functions and their attractive properties. By using *TF*s, the solution of system of Volterra integro-differential equations can be reduced to the solution of algebraic equations. This method can be used to system of integral equations.

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